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Applied to a Curved Surface

By

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On the Singularity at a Concentrated Load Applied to a Curved Surface*

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Summary

This paper deals with the singularity at the point of application of a concentrated load acting perpendicular to a curved boundary of an elastic body. In a neighborhood of the point of application, the boundary is assumed to be representable by a sufficiently smooth arbitrary surface of revolution, the axis of which coincides with the load axis. In the event the surface is locally analytic, it is shown that the singularity is identical with that appropriate to a concentrated load applied normal to a plane boundary if and only if the curvature of the meridian of the surface vanishes at the load point. The required modified singularity for the case of non-vanishing curvature is determined in closed form to the extent where the residual surface tractions are finite and continuous.

Introduction

The singularities encountered in concentrated force problems of the theory of elasticity require special and separate treatment if one is to arrive at practically useful representations of the solution to such problems. Indeed, in order to assure results which are amenable to a complete numerical evaluation, it is essential to determine in closed form the relevant singularities at least to the extent where the residual problem is characterized by finite and continuous surface tractions. This process

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was carried out in a previous paper by F. Rosenthal and one of the present authors [1][1] in connection with the problem of the sphere under concentrated loads. In this particular case it was found that the singularity at a concentrated normal force is not identical with the well known singularity corresponding to a concentrated load applied perpendicular to a plane boundary, and that certain supplementary singularities have to be introduced to effect a reduction of the problem to one obeying the foregoing regularity requirements.

The investigation just cited suggests more general questions: Under what circumstances does Boussinesq's solution [2][2] for the semi-infinite medium bounded by a plane and acted on by a concentrated surface load, supply the complete singularity appropriate to a concentrated load applied to a curved boundary? What are the supplementary singularities needed in the event the Boussinesq singularity fails to yield a regular residual problem? It is the purpose of the present paper to deal with these questions on the assumption that the load is perpendicular to the boundary and that the boundary, in a neighborhood of the load point, is representable by an arbitrary, sufficiently smooth, surface of revolution, the axis of which coincides with the load axis.

The usual conditions imposed on the solution to a problem involving concentrated surface forces, are as follows: (a) it must satisfy the field equations of the theory of elasticity throughout the body under consideration;[3] (b) it must satisfy the boundary conditions for distributed

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] See also [3], p. 191.

[3] It is assumed here that concentrated forces at interior points, and otherwise singular body force distributions, are absent.

tractions;[4] (c) it must possess a singularity at each point of application of a concentrated load such that the resultant of the tractions on any surface enclosing the load point, and lying wholly in the region occupied by the medium, tends to the corresponding prescribed concentrated load in the limit as the surface is contracted to the load point. As emphasized in [1], these three conditions, although necessary, represent an incomplete formulation of the problem and fail to characterize the solution uniquely. A unique characterization is reached by considering the modified problem in which each of the concentrated loads is replaced with an arbitrary continuous distribution of surface tractions over finite surface elements surrounding the load points. The solution to the concentrated force problem is then uniquely defined as the limit of the solution to the modified problem as the surface elements are shrunk to the load points while the resultants of the distributed tractions are made to approach the given concentrated forces.

This definition, which is analogous to Kelvin's definition through a limit process of a concentrated force at an interior point,[5] is natural both on theoretical and on physical grounds; its usefulness ultimately depends on, and is confirmed by, experimental evidence, such as that supplied by Frocht and Guernsey [4]. The solution to a concentrated force problem which is in accord with the definition adopted here, automatically satisfies the three necessary conditions listed earlier. On the other hand, as demonstrated in [1], there exist pseudo-solutions which meet the three conditions cited but fail to agree with the foregoing limit criterion.

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[4]If only concentrated loads are present, the solution must leave the boundary free from surface tractions.

[5]See, for example, [3], art. 130, p. 183.

The validity of Boussinesq's solution for the half-space under a concentrated load is readily established by applying the preceding definition of a concentrated surface force, say, to Cerruti's solution[6] for the semi-infinite body bounded by a plane and subjected to distributed surface tractions. The pseudo-solutions to concentrated force problems discussed in [1] exhibit a common property: their singularities at the load points are of a higher order than that of the Boussinesq singularity. On the other hand, it is clear from similarity considerations that the singularity at a concentrated load applied to a curved surface must remain dominated by the Boussinesq singularity; it cannot be stronger. This observation leads to an additional necessary requirement applicable to the solution of concentrated force problems: (d) the order of the singularity at each load point must not exceed the order r^{-2} of the Boussinesq singularity.[7]

The treatment of the general problem to be considered presently will be based on conditions (a), (b), (c), and (d). In the special instance of the sphere under radial concentrated loads, the correctness of the solution so obtained was verified analytically through a limit process. The question as to whether the four necessary conditions stated always assure a unique solution, which thus coincides with the limit solution defined previously, is apparently not easy to dispose of with complete generality. Until a general affirmative answer is established, solutions to concentrated force problems conforming to the four conditions listed remain subject to individual verification through appropriate limit processes.

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[6]See [3], art. 164-166.

[7]r here is the distance from the load point.

We now turn to the determination of a sequence of singular harmonic functions which is needed in the subsequent analysis. It is hoped that this part of the paper may prove useful beyond the present application.

A Sequence of Harmonic Functions

If (x, y, z) denote Cartesian coordinates, the spherical coordinates (r, θ, γ) are defined through the mapping,

$$\left.\begin{array}{lll} x = r \sin\theta \cos\gamma, & y = r \sin\theta \cos\gamma, & z = r \cos\theta \\ 0 \leq r < \infty, & 0 \leq \theta \leq \pi, & 0 \leq \gamma \leq 2\pi. \end{array}\right\} \qquad (1)$$

With the notation,

$$p = \cos\theta, \qquad \bar{p} = \sin\theta, \qquad (2)$$

we have

$$\rho = (x^2 + y^2)^{1/2} = r\bar{p}, \quad z = rp \qquad (3)$$

for the relations between the spherical coordinates and the cylindrical coordinates (ρ, γ, z). Laplace's equation $\nabla^2 H = 0$, in the axisymmetric case, for which $H = H(r, p)$, becomes

$$H_{rr} + \frac{2}{r} H_r + \frac{\bar{p}^2}{r} H_{pp} - \frac{2p}{r^2} H_p = 0. \qquad (4)^8$$

Equation (4) admits the product solutions,[9]

$$H(r,p) = \left[r^n \text{ or } r^{-n-1}\right]\left[P_n(p) \text{ or } Q_n(p)\right] \quad (n = 0, 1, 2, \ldots), \qquad (5)$$

where P_n and Q_n are the Legendre polynomials and the Legendre functions of the second kind, respectively.

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[8] Subscripts attached to functions which originally bear no subscripts, denote partial differentiation with respect to the argument indicated.

[9] For a comprehensive treatment of spherical harmonics, see [5].

The interior spherical harmonics $r^n P_n(p)$ are regular throughout the finite space whereas the exterior harmonics $r^{-n-1} P_n(p)$ vanish at infinity and, with increasing n, possess progressively stronger singularities at the origin. If we adopt the notation,

$$H_{-n}(r,p) = r^{-n} P_{n-1}(p) \qquad (n = 1, 2, \ldots), \tag{6}$$

then

$$H_{-n-1} = \frac{-1}{n} \frac{\partial}{\partial z} H_{-n}, \tag{7}$$

so that the sequence of exterior harmonics H_{-n} may be generated through successive differentiations with respect to z of the first order exterior harmonic $H_{-1} = r^{-1}$.

For future reference we recall that $P_n(p)$ satisfies Legendre's equation,

$$\frac{d}{dp} (\bar{p}^2 P_n') + n(n + 1) P_n = 0 \qquad (8)[10]$$

and cite the recursion formulas,

$$\left.\begin{aligned} P_{-n-1} &= P_n \\ (2n + 1)\, pP_n &= (n + 1)\, P_{n+1} + nP_{n-1} \\ \bar{p}^2 P_n' &= nP_{n-1} - npP_n . \end{aligned}\right\} \tag{9}$$

Furthermore, we record the special values

$$\left.\begin{aligned} P_{2n}(0) &= (-1)^n \frac{\underline{|2n}}{2^{2n}(\underline{|n})^2}, \qquad & P_{2n+1}(0) &= 0; \\ P_{2n}'(0) &= 0, & P_{2n+1}'(0) &= (2n+1)\, P_{2n}(0) \end{aligned}\right\} \tag{10}$$

and

$$P_n(1) = 1, \qquad P_n'(1) = \frac{n(n+1)}{2}. \tag{11}$$

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[10] The argument of P_n is henceforth assumed to be p and $P_n' \equiv \frac{dP_n}{dp}$.

The Legendre functions of the second kind admit the representation,

$$Q_n(p) = \frac{1}{2} P_n(p) \ln \frac{1+p}{1-p} - W_{n-1}(p) \qquad (n = 0, 1, 2, \ldots) \tag{12}$$

in which

$$\left.\begin{aligned} W_{n-1} &= \frac{2n-1}{1\cdot n} P_{n-1} + \frac{2n-5}{3\cdot(n-1)} P_{n-3} + \frac{2n-9}{5\cdot(n-2)} P_{n-5} + \cdots \\ &= \sum_{m=1}^{n} \frac{1}{m} P_{m-1} P_{n-m}, \qquad W_{-1} = 0. \end{aligned}\right\} \tag{13}$$

Since $Q_n(p)$ has a logarithmic singularity at $p = \pm 1$, the harmonics $r^n Q_n(p)$ and $r^{-n-1} Q_n(p)$ in (5) are singular along the entire z-axis.

In the problem under consideration we shall have need for a sequence of axisymmetric harmonic functions which is regular in the half-space $z \geq 0$ with the exception of the origin $r = 0$, where it shall possess singularities, progressively weaker than that of $H_{-1} = r^{-1}$. Such a set of potentials may be constructed through successive integrations with respect to z of the first order exterior harmonic H_{-1}. We thus seek a sequence of functions $H_n(r,p)$ which satisfies the following requirements:

$$\nabla^2 H_n = 0, \tag{14}$$

$$\frac{\partial}{\partial z} H_n = n H_{n-1} \quad (n = 1,2,\ldots), \qquad \frac{\partial}{\partial z} H_o = H_{-1} = \frac{1}{r}. \tag{15}$$ [11]

Through a process of induction we are led to

$$\left.\begin{aligned} H_n(r,p) &= r^n T_n(p) \qquad (n = 0, 1, 2, \ldots) \\ T_n(p) &= P_n(p) L - V_n(p), \end{aligned}\right\} \tag{16}$$

where

$$L = \ln r\,(1+p) \equiv \ln (r+z), \tag{17}$$

[11] The coefficient n in the first of (15) is introduced for later convenience. Note that (15) is analogous to (7).

and

$$\left.\begin{aligned} V_n &= (3c_n - 2c_{2n})\, P_n - 2 \sum_{m=0}^{n-1} \frac{(-1)^{n-m}\,(2m+1)\, P_m}{(n-m)(n+m+1)}\,, \\ V_o &= 0, \end{aligned}\right\}(18)$$

provided,

$$c_n = \sum_{m=1}^{n} \frac{1}{m} \qquad (n = 1,\, 2,\, \ldots). \tag{19}$$

In order to verify that the functions H_n, defined by (16) to (19) have the properties (14) and (15), we first observe that a lengthy computation, based on (18), (19), and (9), yields the recursion relations,

$$\left.\begin{aligned} \bar{p}^2\, V_n' + npV_n - nV_{n-1} &= P_n, \\ n(nV_n - pV_n' + V_{n-1}') &= P_n' - 2P_{n-1}', \\ &(n = 1,\, 2,\, \ldots), \end{aligned}\right\}(20)$$

which, in turn, imply the differential equation,

$$\frac{d}{dp}\,(\bar{p}^2\, V_n') + n(n+1)V_n = 2(P_n' - P_{n-1}') \qquad (n = 0,1,2,\ldots). \tag{21}$$

From (16), (20), (21), (9), we have

$$\left.\begin{aligned} \bar{p}^2 T_n' + npT_n - nT_{n-1} &= -\, pP_n, \\ n(nT_n - pT_n' + T_{n-1}') &= 2P_{n-1}' - pP_n', \\ &(n = 1,\, 2,\, \ldots), \end{aligned}\right\}(22)^{12}$$

$$\frac{\partial}{\partial p}\,(\bar{p}^2 T_n') + n(n+1)T_n = -\,(2n+1)P_n, \qquad (n = 0,1,2,\ldots). \tag{23}$$

With the aid of (16), (22), (23), and (4), Equations (14), (15) now readily follow. For future purposes we list the special values,

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[12] Note that $T_n' \equiv \dfrac{\partial T_n}{\partial p}$.

$$\left.\begin{aligned} V_{2n}(0) &= c_n P_{2n}(0), & V_{2n+1}(0) &= \Big[(2n+1)P_{2n}(0)\Big]^{-1} \\ V'_{2n}(0) &= P_{2n}(0) - \Big[P_{2n}(0)\Big]^{-1}, & V'_{2n+1}(0) &= (2n+1)c_n P_{2n}(0), \end{aligned}\right\} \quad (24)$$

which are obtained by induction with the aid of (18), (19), (22), (9), and (10).

According to (12), (13), (18),

$$V_n(p) - (-1)^n \, V_n(-p) = 2W_{n-1}(p) \quad (25)$$

and (12), (16), (17), (25) yield

$$H_n(r,p) - (-1)^n \, H_n(r,-p) = 2r^n \, Q_n(p). \quad (26)$$

This establishes the connection between the harmonic functions H_n, which are no longer product solutions of Laplace's equation, and the spherical harmonics of the second kind given in (5). Finally, we record explicitly the first three members of the aggregate $H_n(r,p)$:

$$\left.\begin{aligned} H_o &= \ln r(1+p), \qquad H_1 = r\Big[p \ln r(1+p) - 1\Big], \\ H_2 &= \frac{r^2}{2}\Big[(3p^2-1)\ln r(1+p) - p^2 - 3p + 1\Big]. \end{aligned}\right\} \quad (27)$$

Sequences of Singular Solutions

The sequence of harmonic functions established previously may be used to generate certain sequences of singular solutions of the field equations of elasticity theory. In the absence of body forces, and in case of torsionless axisymmetry about the z-axis, the general solution of the displacement equations of equilibrium may be written as,[13]

$$2G\Big[u_\rho, \, u_\gamma, \, u_z\Big] = \text{grad}\,(\phi + z\psi) - \Big[0, 0, 4(1-\nu)\psi\Big] \quad (28)$$

and

$$\nabla^2\phi(\rho,z) = \nabla^2\psi(\rho,z) = 0, \quad (29)$$

[13]See [2], [6].

where $\left[u_\rho, u_\gamma, u_z\right]$ are the cylindrical components of displacement, whereas G and ν designate the shear modulus and Poisson's ratio, respectively.

We define two sequences of particular solutions[14] $[A_n]$ and $[B_n]$ in terms of their generating stress functions as follows:

$$\left.\begin{aligned} &[A_n] \ldots \phi = H_n(r,p), && \psi = 0 && (n = 0,1,2,\ldots) \\ &[B_n] \ldots \phi = 0, && \psi = H_n(r,p) && (n = 0,1,2,\ldots) \end{aligned}\right\} \quad (30)$$

in which H_n is defined by (16) to (19). The spherical components of displacement u_r, u_θ and the spherical stress components σ_r, σ_θ, σ_γ, $\tau_{r\theta}$, belonging to (28), (29), were given explicitly in [7].[15] Substitution of (30) into (4) to (7) of Reference [7], and use of (22), (23), (8), (9), yields the following representations of $[A_n]$, $[B_n]$ in spherical coordinates.

$[A_n]$:

$$\left.\begin{aligned} &2Gu_r = r^{n-1}(nT_n + P_n), \qquad 2Gu_\theta = -\bar{p}\, r^{n-1}\, T_n' \\ &\sigma_r = r^{n-2}\left[n(n-1)T_n + (2n-1)P_n\right] \\ &\sigma_\theta = r^{n-2}\left[-n^2 T_n + pT_n' - 2nP_n\right] \\ &\sigma_\gamma = r^{n-2}\left[nT_n - pT_n' + P_n\right] \\ &\tau_{r\theta} = -\bar{p}\, r^{n-2}\left[(n-1)T_n' + P_n'\right] \end{aligned}\right\} \quad (31)$$

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[14] Throughout this paper capital letters in brackets denote either the displacement vector-field or the stress tensor-field of a solution of the field equations, and equality, addition, as well as multiplication by a scalar, are to be interpreted accordingly.

[15] u_γ, $\tau_{\gamma r}$, $\tau_{\gamma\theta}$ vanish identically in view of the assumed rotational symmetry.

$[B_n]$:

$$\left.\begin{aligned}
2Gu_r &= pr^n\left[(n-3+4\nu)\,T_n + P_n\right]\\
2Gu_\theta &= \bar{p}r^n\left[(3-4\nu)\,T_n - pT_n'\right]\\
\sigma_r &= r^{n-1}\left[n(n-3+2\nu)pT_n + (2n-3+2\nu)pP_n - 2\nu\bar{p}^{-2}T_n'\right]\\
\sigma_\theta &= r^{n-1}\left\{-n(n+2\nu)pT_n - 2(n+\nu)pP_n + \left[1-(3-2\nu)\bar{p}^{-2}\right]T_n'\right\}\\
\sigma_y &= r^{n-1}\left\{(1-2\nu)p(nT_n+P_n) - \left[1-(1-2\nu)\bar{p}^{-2}\right]T_n'\right\}\\
\tau_{r\theta} &= \bar{p}r^{n-1}\left[(1-2\nu)(nT_n+P_n) - (n-2+2\nu)pT_n' - pP_n'\right]
\end{aligned}\right\}\tag{32}$$

Let $Z(r)$ be the resultant force of the tractions to which a stress field with rotational symmetry about the z-axis gives rise on a hemisphere centered at $r = 0$, lying wholly in the region $z \geq 0$, and having an outer normal which is directed toward the origin. If $Z(r)$ is positive when its sense is that of the positive z-axis, it was shown in [8][16] that

$$Z(r) = 2\pi r\left\{\left[\frac{\partial\phi}{\partial r}\right]_{p=0} + 2(1-\nu)r\int_0^1 \frac{\partial\psi}{\partial r}\,dp\right\}, \tag{33}$$

where $\phi(r,p)$ and $\psi(r,p)$ are the corresponding generating stress functions. Applying (33) to (30), we find, with the aid of (9), (10), (11), (16), (17), (23), and (24), the values of $Z(r)$ for $[A_n]$ and $[B_n]$.

$$\left.\begin{aligned}
[A_{2n}]:&\quad Z(r) = \frac{2\pi(-1)^n \lfloor 2n \; r^{2n}}{2^{2n}(\lfloor n)^2}\left[2n\ln r + 1 - 2nc_n\right]\\
[A_{2n+1}]:&\quad Z(r) = -\frac{2\pi(-1)^n\, 2^{2n}(\lfloor n)^2\, r^{2n+1}}{\lfloor 2n}
\end{aligned}\right\}\tag{34}$$

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[16] See Equations (31), (32) of Reference [8].

$$\left.\begin{aligned}
[B_{2n}]: \quad & Z(r) = \frac{4\pi(1-\nu)(-1)^n\, 2^{2n}\, (\lfloor n)^2\, r^{2n+1}}{\lfloor 2n+1} \\
[B_{2n-1}]: \quad & Z(r) = \frac{-4\pi(1-\nu)(-1)^n \lfloor 2n+1\, r^{2n}}{2^{2n}\, (\lfloor n)^2}\left[2n \ln r + 1 - 2nc_n\right]
\end{aligned}\right\} \quad \begin{matrix}(34)\\(\text{Cont.})\end{matrix}$$

It follows from (34) that the aggregate of solutions defined by the linear combination,

$$\left.\begin{aligned}
&[E_n] = 2(1-\nu)[A_{n+1}] + (n+1)[B_n] \\
&(n = 0, 1, 2, \ldots)
\end{aligned}\right\} \quad (35)$$

is self-equilibrated in the sense that $Z(r) = 0$ for $[E_n]$. For later convenience, we record here the displacements and stresses of the first two members of the sequence $[E_n]$, which also appear in [1].

$$\left.\begin{aligned}
&[E_o]: \\
&2Gu_r = -(1-2\nu)pL + (3-2\nu)p - 2(1-\nu) \\
&2Gu_\theta = \bar{p}\left[(1-2\nu)L - (3-2\nu)\frac{p}{1+p}\right] \\
&\sigma_r = -\frac{1}{r}\left[(1-2\nu)p + 2\nu\right] \\
&\sigma_\theta = \frac{1}{r(1+p)}\left[(1-2\nu)p^2 - 2p\right] \\
&\sigma_y = \frac{1}{r(1+p)}\left[(1-2\nu)p - 2\right] \\
&\tau_{r\theta} = \frac{\bar{p}}{r(1+p)}\left[(1-2\nu)p - 1\right]
\end{aligned}\right\} \quad (36)$$

$$\left.\begin{aligned}
&[E_1]: \\
&2Gu_r = r\left\{2\left[\nu(1+p^2) - \bar{p}^2\right]L + (3-\nu)p^2 - 2(1+\nu)p + 1 - \nu\right\} \\
&2Gu_\theta = \bar{p}r\left\{-2(1+\nu)pL + \frac{1}{1+p}\left[(-3+\nu)p^2 - (1-3\nu)p - 2(1-2\nu)\right]\right\} \\
&\sigma_r = -2(1+\nu)(\bar{p}^2L + p) + (5+\nu)p^2 - 1 + \nu \\
&\sigma_\theta = -2(1+\nu)p^2L + \frac{p}{1+p}\left[-(5+\nu)p^2 - (3-\nu)p + 6\right] \\
&\sigma_y = -2(1+\nu)L + \frac{1}{1+p}\left[(-3+\nu)p + 1 - \nu\right] \\
&\tau_{r\theta} = -2(1+\nu)p\bar{p}L + \frac{\bar{p}}{1+p}\left[-(5+\nu)p^2 - (3-\nu)p + 2\right]
\end{aligned}\right\} \quad (37)$$

Solutions $[A_n]$, $[B_n]$, $[E_n]$ $(n = 0, 1, 2, \ldots)$, with increasing n, possess progressively weaker singularities along the portion $z \leq 0$ of the z-axis, and are otherwise regular in the finite space; for $n = 0$ these solutions vanish as $z \to \infty$ so far as the stresses are concerned.

Construction of the Singularity due to a Concentrated Surface Force on a Curved Surface

Boussinesq's solution [2][17] corresponding to a concentrated load of magnitude $|Q|$ applied at $r = 0$ normal to the plane boundary of a medium occupying the half-space $z \geq 0$, appears as

$$[S_o] = -\frac{Q}{2\pi}\left\{(1-2\nu)[A_o] + [B_{-1}]\right\}, \qquad (38)^{18}$$

where $[A_o]$ is defined by (30) and

$$[B_{-1}] \ldots \phi = 0, \quad \psi = H_{-1} = r^{-1}. \qquad (39)$$

The displacements and stresses of $[S_o]$ follow.

$$\left.\begin{aligned}
&[S_o]: \\
&u_r = \frac{Q}{4\pi G r}\left[4(1-\nu)p - 1 + 2\nu\right] \\
&u_\theta = \frac{Q\bar{p}}{4\pi G r}\left[\frac{1-2\nu}{1+p} - 3 + 4\nu\right] \\
&\sigma_r = \frac{-Q}{2\pi r^2}\left[2(2-\nu)p - 1 + 2\nu\right], \qquad \sigma_\theta = \frac{Q(1-2\nu)p^2}{2\pi r^2(1+p)} \\
&\sigma_y = \frac{Q(1-2\nu)(p^2+p-1)}{2\pi r^2(1+p)}, \qquad \tau_{r\theta} = \frac{Q(1-2\nu)p\bar{p}}{2\pi r^2(1+p)}
\end{aligned}\right\} (40)$$

Evidently, $Z(r) = Q$ for $[S_o]$, as is readily verified by means of (38), (39), (30), and (33).

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[17] See also [3], p. 191.

[18] If $Q > 0$ the force has the sense of the positive z-axis.

We now turn to a concentrated load acting perpendicular to a curved boundary (Figure 1). Let the point of application of the load of magnitude $|Q|$ be the origin and again let $Q > 0$ if the load acts in the positive z-direction. Suppose that the boundary, in a neighborhood of the origin, coincides with a surface of revolution Σ, the semi-meridian Γ of which admits the representation,

$$\left.\begin{aligned} \Gamma:\ \ z = f(\rho) \qquad & (0 \leq \rho \leq \epsilon) \\ f(0) = f'(0) = 0, & \end{aligned}\right\} \quad (41)$$

and let $f(\rho)$ be at least four times continuously differentiable.[19] Since $f'(0) = 0$, Σ possesses a uniquely defined tangent plane[20] at its vertex $r = 0$. The function $f(\rho)$ has a Taylor expansion of the form,

$$f(\rho) = a_2\rho^2 + a_3\rho^3 + \dots \qquad (42)$$

so that

$$a_2 = \frac{1}{2} f''(0) = \frac{k_o}{2}, \qquad a_3 = \frac{1}{6} f'''(0) = \frac{k_o'}{6} \qquad (43)$$

where $k(\rho)$ is the curvature of Γ and

$$k_o = k(0), \qquad k_o' = \left[\frac{dk}{d\rho}\right]_{\rho=0}. \qquad (44)$$

With a view toward examining the nature of the singularity at the load point $r = 0$, we first determine the normal and shearing tractions σ and τ (Figure 1) induced by the Boussinesq solution $[S_o]$ on the arc Γ. To this end we observe by means of the law of stress transformation and elementary geometric considerations that along Γ,

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[19]Note that the regularity restrictions on f imply the same degree of smoothness for Σ only if the (right-hand) derivatives of odd order of $f(\rho)$ at $\rho = 0$ vanish. Otherwise, the even extension of $f(\rho)$ exhibits discontinuities in its derivatives of odd order at the origin.

[20]The subsequent analysis is not valid if $r = 0$ is a conical point of Σ.

$$\left.\begin{aligned}\sigma &= \frac{\sigma_r + \sigma_\theta}{2} + \frac{\sigma_r - \sigma_\theta}{2} \cos 2(\beta + \theta) - \tau_{r\theta} \sin 2(\beta + \theta)\\ \tau &= \frac{\sigma_r - \sigma_\theta}{2} \sin 2(\beta + \theta) + \tau_{r\theta} \cos 2(\beta + \theta)\end{aligned}\right\} \quad (45)$$

in which β is the inclination of the tangent of Γ. Evidently,

$$\cos 2\beta = \frac{1 - (f')^2}{1 + (f')^2}, \qquad \sin 2\beta = \frac{2f'}{1 + (f')^2}. \quad (46)$$

Recalling that

$$r = \rho \left[1 + (\frac{z}{\rho})^2\right]^{1/2}, \quad (47)$$

we reach by means of (3), (41), (42), the expansions,

$$\left.\begin{aligned}\frac{1}{r} &= \frac{1}{\rho} (1 - \frac{a_2^2}{2} \rho^2 - a_2 a_3 \rho^3 + \ldots)\\ p &= a_2\rho + a_3\rho^2 + (a_4 - \frac{a_2^3}{2}) \rho^3 + \ldots\\ \bar{p} &= 1 - \frac{a_2^2}{2} \rho^2 - a_2 a_3 \rho^3 + \ldots,\end{aligned}\right\} \quad (48)$$

valid on Γ. From (46), (48), and (2) follows

$$\left.\begin{aligned}\cos 2(\beta + \theta) &= - 1 + 2a_2^2\rho^2 + 8a_2 a_3\rho^3 + \ldots\\ \sin 2(\beta + \theta) &= - 2a_2\rho - 4a_3\rho^2 - 6(a_4 - a_2^3)\rho^3 + \ldots.\end{aligned}\right\} \quad (49)$$

Substitution of (49) and (40) into (45), after a lengthy routine computation, and final use of (43), yields

$$\left.\begin{aligned}&[S_o]:\\ &\sigma(\rho) = \frac{Q(1 - 2\nu)\, k_o^2}{2\pi} + o(1) = O(1)\\ &\tau(\rho) = \frac{Q}{8\pi}\left[\frac{-4(1 - 2\nu)k_o}{\rho} + (5 - 4\nu)k_o^2 - 2(1 - 2\nu)k_o'\right] + o(1)\end{aligned}\right\} \quad (50)$$

where $o(1)$ and $O(1)$, respectively, denote functions of ρ which tend to zero or to a finite limit as ρ approaches zero.

If $[S]$ designates the complete solution to the concentrated force problem under consideration, we may write,

$$[S] = [S_o] + [R_o], \tag{51}$$

in which $[R_o]$ stands for the solution to a "residual problem", the characterization of which is implicit in (51). We shall suppose, without loss in generality, that the boundary of the medium is free from distributed tractions on Σ. Furthermore, we shall agree to call $[R_o]$ "regular" on Σ if the residual problem on Σ is governed by finite and continuous surface tractions. This requires for $[R_o]$,

$$\sigma(\rho) = O(1), \qquad \tau(\rho) = o(1). \tag{52}$$ [21]

Equations (50), (51) now permit the following conclusions. $[R_o]$ is regular on Σ if and only if the curvature k and the rate of change of curvature k' of the semi-meridian Γ of Σ both vanish at the load point. We note parenthetically that k_o' is zero automatically if the even extension of Γ has at least a continuous rate of change of curvature.[22] In the event that the even extension of $f(\rho)$ is an analytic function, so that Σ is an analytic surface, the surface tractions of $[R_o]$ are analytic on Σ if and only if $k_o = 0$. In this case, and in this case only, the Boussinesq singularity constitutes the entire singularity of $[S]$ at the load point in question.

In order to effect a reduction of the problem to a residual problem which is regular on Σ when k_o and k_o' are not both zero, we need to introduce supplementary singular solutions, in addition to $[S_o]$. Moreover, in accordance with conditions (c) and (d) stated in the Introduction, the

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[21] Observe that $\tau(\rho) = O(1)$ is insufficient since it admits a discontinuity in the surface shearing tractions of $[R_o]$ at the vertex of Σ.

[22] See footnote No. 19.

supplementary singularities sought must be self-equilibrated and of a lower order than r^{-2}. This leads us to solutions $[E_n]$ defined by (35) and listed explicitly for $n = 0, 1$ in (36), (37).

The local behavior of $[E_o]$, $[E_1]$ on Γ, in the vicinity of $\rho = 0$, is established by a process which is strictly analogous to that followed in the derivation of (50). We may, therefore, give the results of these computations directly.

$$\left.\begin{array}{l} [E_o]: \\ \sigma(\rho) = -2k_o + o(1), \qquad \tau(\rho) = \dfrac{1}{\rho} - (1 - 2\nu)k_o + o(1) \end{array}\right\} \quad (53)$$

$$\left.\begin{array}{l} [E_1]: \\ \sigma(\rho) = o(1), \qquad \tau(\rho) = -2 + o(1) \end{array}\right\} \quad (54)$$

It is apparent from (53), (54) that τ of $[E_o]$ still becomes infinite at $\rho = 0$ whereas τ of $[E_1]$ displays merely a finite discontinuity at the origin; the normal tractions σ of both of these solutions are again finite and continuous on Σ, as in the case of $[S_o]$.

We now set

$$\left.\begin{array}{l} [S] = [S_2] + [R_2] \\ [S_2] = [S_o] + \lambda_o[E_o] + \lambda_1[E_1], \end{array}\right\} \quad (55)$$

and ask for the values of the coefficients λ_o, λ_1 which assure the regularity in the sense of (52) of the solution $[R_2]$ to the new residual problem. In view of (50), (53), (54) this leads to

$$\left.\begin{array}{l} \lambda_o = \dfrac{(1 - 2\nu)\, Qk_o}{2\pi} \\ \lambda_1 = \dfrac{Q}{16\pi}\left[(1 + 12\nu - 16\nu^2)k_o^2 - 2(1 - 2\nu)k_o'\right] \end{array}\right\} \quad (56)$$

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With $[S_2]$ determined as in (55), (56), the surface tractions of $[R_2]$ are finite and continuous on Σ. Alternatively, $[S]$ admits the representation,

$$\left.\begin{aligned}[S] &= [S_1] + [R_1]\\ [S_1] &= [S_o] + \lambda_o [E_o],\end{aligned}\right\} \quad (57)$$

where λ_o again has the value appearing in (56). The surface tractions of $[R_1]$ are finite but exhibit a discontinuity in τ at the origin.

It is important to emphasize that the surface tractions of $[R_2]$ on Σ, although finite and continuous, are by no means analytic even if Σ is analytic. This is due to the fact that the derivatives with respect to ρ of σ and τ in $[E_1]$, unlike the corresponding derivatives belonging to $[S_o]$, $[E_o]$, possess a logarithmic singularity[23] at $\rho = 0$. Consequently, the singularity of $[S_2]$, regardless of the analyticity of Σ, does not represent the entire singularity inherent in $[S]$ at the load point.

The sequence of singular solutions $[S_o]$, $[S_1]$, $[S_2]$ thus gives rise to a sequence of solutions $[R_o]$, $[R_1]$, $[R_2]$ to the complementary residual problems, which are characterized by progressively increasing regularity in the corresponding boundary conditions. The process of successive "regularization" of the residual problem can be continued in an obvious manner by recourse to the remaining members of the sequence of self-equilibrated singular solutions $[E_n]$ defined in (35).

If Σ is a portion of a sphere of radius r_o, lying in the half-space $z \geq 0$, then $k_o = r_o^{-1}$, $k_o' = 0$, and the results obtained here reduce to those given in [1] in connection with the particular problem of the

[23]The functions designated by $o(1)$ in (54) contain a term, $g(\rho) = p \ln r(1 + p)$.

sphere under concentrated loads. Whereas the general questions raised in the Introduction have been answered within the limitations placed on the relative orientation of the load and on the local character of the boundary, it should be pointed out that certain difficulties in the large may arise in the application of the results to specific concentrated force problems. Thus, the representations (51), (55), (57) lead to serious complications if the line of action of a concentrated load intersects the boundary of the medium on opposite sides of the tangent plane at the point of application of the load. Here $[S_o]$, $[S_1]$, $[S_2]$, each of which is singular along the negative z-axis, give rise to line singularities in the interior of the body.[24] Finally, if the region occupied by the medium is not bounded, the representation (55) becomes inadmissible in view of the behavior of $[E_1]$ at infinity.

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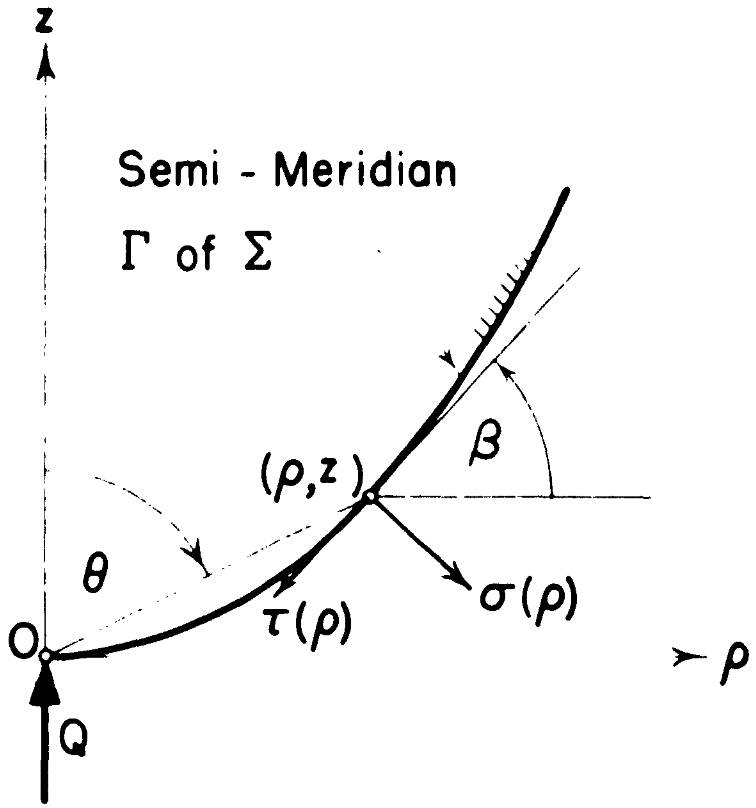

Figure 1.

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